

# REAL SURFACES IN ELLIPTIC SURFACES

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**ABSTRACT.** We study the structure of complex points on real surfaces, embedded into complex Elliptic surfaces. We show, for example, that any compact surface has a totally real embedding into a blow-up of a  $K3$  surface. We also exhibit smooth disc bundles over compact orientable surfaces that have a Stein structure as Stein domains inside Elliptic surfaces.

## 1. STATEMENT OF RESULTS

Let  $S$  be a real surface, embedded into a complex surface  $(X, J)$ . We say that the embedding  $S \hookrightarrow X$  is *totally real* at a point  $p \in S$ , if  $T_p S + J(T_p S) = T_p X$ . If this is not the case, we call  $p$  a *complex point* of the embedding. An embedding is called totally real, if it is totally real at all points.

**Theorem 1.1.** *Every compact oriented real surface  $S$  has a totally real embedding into any  $K3$  surface. Every compact real surface has a totally real embedding into a blow-up of a  $K3$  surface at one point.*

A blow-up of a complex surface is of course not minimal. If we want to have an embedding of all compact surfaces into a minimal surface, we have the following theorem.

**Theorem 1.2.** *Every compact real surface has a totally real embedding into any  $E(3)$  surface.*

Let us denote by  $\Sigma_g$  the compact Riemann surface of genus  $g \geq 0$ , and let  $n$  be an integer. We denote by  $D(g, n)$  the open unit disc bundle over the surface  $\Sigma_g$ , with Euler number  $n$ . It follows from the adjunction inequality for Stein surfaces that for  $n > 2g - 2$ , the smooth manifolds  $D(g, n)$  do not have any Stein structure. It is furthermore a consequence of the result of Gompf [8], using a method of Stein surgery developed by Eliashberg in [2], that for  $n \leq 2g - 2$ , the smooth manifolds  $D(g, n)$  can be endowed with a Stein structure. We use a method of Stein fattening, introduced by Forstnerič [5], to give a different proof of this result, by explicitly seeing  $D(n, g)$  as open strictly pseudoconvex Stein domains in Elliptic surfaces  $E(n)$ . The definition of Elliptic surfaces  $E(n)$  is given latter in the text.

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**Theorem 1.3.** *For  $n \leq 2g - 2$ , the smooth manifold  $D(n, g)$  can be realized as an open strictly pseudoxconvex Stein domain in a complex Elliptic surface  $E(-n + 2g)$ .*

For compact nonorientable surfaces, the situation is even less rigid as in Theorem 1.3. Let us denote  $\tilde{D}(n, \chi)$  the disc bundle over a compact nonorientable surface with Euler characteristic  $\chi$  and Euler number  $n$ . By passing to a double cover and using adjunction inequality for Stein surfaces,  $\tilde{D}(n, \chi)$  can have a Stein structure only if  $n + \chi \leq 0$ . See [7]. We can construct such Stein structures using the following theorem.

**Theorem 1.4.** *For  $n + \chi \leq 0$ , the smooth manifold  $\tilde{D}(n, \chi)$  can be realized as an open strictly pseudoxconvex Stein domain in sufficiently high blow-up of  $\mathbb{C}P^2$ .*

## 2. COMPLEX POINTS OF REAL SURFACES IN COMPLEX SURFACES

Let  $S$  be a real compact surface in a complex surface  $X$ . After perhaps a small generic perturbation, the embedding  $S \hookrightarrow X$  is totally real outside a finite collection of complex points, which can, following Bishop [1], be classified as either *elliptic* or *hyperbolic*. This means that, locally around a complex point  $p$ , we can choose complex coordinates  $(z, w)$  on  $X$ , so that  $S$  is written as

$$w = \alpha z \bar{z} + \frac{1}{2} z^2 + \frac{1}{2} \bar{z}^2 + o(|z|^3),$$

where  $\alpha \in [0, \infty]$  is a holomorphic invariant of the complex point (the case  $\alpha = \infty$  should be understood as the surface  $w = z \bar{z} + o(|z|^3)$ ). Elliptic points correspond to  $\alpha > 1$ , hyperbolic to  $\alpha < 1$  and (nongeneric) parabolic to  $\alpha = 1$ .

Let  $S \hookrightarrow X$  be a generic embedding of a compact surface with only finitely many complex points, all of them either elliptic or hyperbolic. Let  $e(S)$  be the number of elliptic complex points on  $S$ , and  $h(S)$  a number of hyperbolic complex points on  $S$ . The algebraic number of complex points,  $I(S) = e(S) - h(S)$ , is a topological invariant of the embedding, and can be expressed as

$$(2.1) \quad I(S) = \chi(S) + \chi(NS),$$

where  $\chi(S)$  is the Euler characteristic of  $S$  and  $\chi(NS)$  is the Euler characteristic of the normal bundle  $NS$  of the embedding, which can be calculated as the self-intersection number  $[S]^2$  in the case of an orientable  $S$ . For the proof, see [17]. In the case of an oriented surface  $S \hookrightarrow X$ , one can also talk about a sign at a complex point  $p \in S$ : positive, if the orientation of the tangent space  $T_p S$  agrees with the induced orientation on  $T_p S$  as a complex subspace of  $T_p X$ , and negative otherwise. The signed algebraic numbers of complex points,  $I_{\pm}(S) = e_{\pm}(S) - h_{\pm}(S)$ , also turn out to be topological invariants. Here the sign in the subscripts indicate the sign of complex points.

They can be expressed by Lai formulae [12]

$$(2.2) \quad 2I_{\pm}(S) = \chi(S) \pm \langle c_1(X), [S] \rangle + [S]^2,$$

where  $c_1(X) = c_1(TX)$  is the first Chern class of the complex surface  $X$ .

By a result of Harlamov and Eliashberg [3], and more generally Forstneric [5], a pair of elliptic and hyperbolic complex points can always be cancelled on nonorientable surfaces, and can be cancelled on orientable surfaces exactly when the signs at the complex points agree. The cancellation can be done by a  $C^0$  small isotopy, changing the embedding only in an arbitrary small neighborhood of an arc, connecting the two complex points.

By a result of Bishop [1], a surface  $S \hookrightarrow X$  having elliptic complex points can never have a Stein neighborhood basis in  $X$ , since the local hull of holomorphy is nontrivial at elliptic complex points. Up to changing an embedding by an isotopy, this is the only obstruction to having a Stein basis: if  $S$  is a compact oriented surface in a complex surface  $X$  with  $I_{\pm}(S) \leq 0$ , then  $S$  is  $C^0$  isotopic to a surface having a regular Stein neighborhood basis, see [5]. The same result holds if the surface  $S$  is compact and nonorientable, with the topological condition now being  $I(S) \leq 0$ . This is most easily proven by first cancelling all the elliptic points by an isotopy, and then by further isotoping the surface, putting the remaining hyperbolic points to be of a special type, so that the usual distance function gives a regular Stein neighborhood basis. For a careful proof, see [5, 6]. By a regular basis of  $S$ , we mean a basis system of open neighborhoods  $\{U_{\epsilon}\}_{0 \leq \epsilon \leq 1}$  of  $S$  in  $X$ , satisfying

- $\Omega_{\epsilon} = \bigcup_{s < \epsilon} \Omega_s$ ,
- $\bar{\Omega}_{\epsilon} = \bigcap_{s > \epsilon} \Omega_s$ ,
- $S = \bigcap_{s > 0} \Omega_s$  is a strong deformation retract of  $\Omega_{\epsilon}$ .

Up to first performing a small isotopy of a surface, finding compact surfaces with regular Stein neighborhood basis reduces to checking whether  $I_{\pm}$  (or  $I$  if the surface is unorientable) is nonpositive for the given embedding. It turns out that this is very often automatically the case. Using Seiberg-Witten theory, Kronheimer-Mrowka [11], Fintushel-Stern [4] and Ozsvath-Szabo [16], have proved adjunction inequalities for surfaces in many 4-manifolds, implying that  $I_{\pm}(S) \leq 0$  for any oriented surface imbedded in a compact Kähler surface with  $b_2^+ > 1$ , except for the embedded spheres, where one has a slightly weaker inequality,  $I_{\pm} \leq 2$ . This led to the result of Lisca-Matić [13], stating that  $I_{\pm}(S) \leq 0$  holds for all oriented real surfaces embedded into Stein surfaces, except for homologically trivial spheres. We call this result the adjunction inequality for Stein surfaces.

### 3. ELLIPTIC SURFACES

**Definition 3.1.** A complex elliptic surface is a compact complex surface  $X$ , together with a holomorphic map  $f: X \rightarrow C$ , where  $C$  is a compact complex curve, so that all but finitely many fibers  $f^{-1}(z)$  are elliptic curves.

Here we only study a special kind of Elliptic surfaces, called  $E(n)$  surfaces. Up to diffeomorphism, they can be given by the following construction.

Let  $\mathbb{CP}^2 \rightarrow \mathbb{CP}^1$  be a pencil of cubics over  $\mathbb{CP}^1$ . By this, we mean a collection of curves  $\{z_0p_0 + z_1p_1, [z_0, z_1] \in \mathbb{CP}^1\}$  in  $\mathbb{CP}^2$ , where  $p_0, p_1$  are generic (intersecting at 9 distinct points) cubics in  $\mathbb{CP}^2$ . Let  $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \rightarrow \mathbb{CP}^1$  be the fibration gotten by blowing up at the 9 singularities. The generic fiber of this fibration is an elliptic curve, but since  $\chi(E(1)) = 12$ , we must also have some singular fibres. To construct surfaces  $E(n)$  we need the fibre sum operation. Let us assume we have constructed  $E(1), \dots, E(n-1)$ . Let  $F_1$  be a regular fibre of  $E(1) \rightarrow \mathbb{CP}^1$  and let  $F_2$  be a regular fibre of  $E(n-1) \rightarrow \mathbb{CP}^1$ . Let  $U_1$  and  $U_2$  be tubular neighborhoods of  $F_1$  and  $F_2$  respectively, and let  $\phi: \partial U_1 \rightarrow \partial U_2$  be fibre preserving, orientation reversing diffeomorphism. Then  $E(n) = E(1) \#_f E(n-1) := (E(1) \setminus U_1 \cup E(n-1) \setminus U_2) / \phi$ .

We gave a definition of  $E(n)$ , usually found in the literature. It is very convenient for calculating properties of  $E(n)$ , but from it, it is not obvious that the diffeomorphism in the fibre sum operation can be chosen so the surfaces  $E(n)$  have a complex structure. This is indeed the case, as one can see from alternative descriptions of surfaces  $E(n)$ . For a more algebraic treatment of Elliptic surfaces, see for example [9].

We now list some homological properties of Elliptic surfaces  $E(n)$ . The results are classical, and can be found in [7].

**Proposition 3.2.** *The 4-manifold  $E(n)$  is simply connected with  $H_2(E(n), \mathbb{Z}) = \mathbb{Z}^{12n-2}$  and intersection form*

$$n(-E_8) \oplus 2(n-1) \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & -n \end{bmatrix},$$

where all basis elements of  $H_2(E(n), \mathbb{Z})$  are represented by embedded spheres, except the elements with self intersection 0, which can be represented by embedded by tori. The homology element with self intersection  $-n$  is the class of a section (sphere) of the fibration.

**Proposition 3.3.** *For any  $E(n)$  surface we have  $c_1(E_n) = (2-n)PD(f)$ , where  $PD(f)$  is the Poincare dual of the class of a fiber of the elliptic fibration  $E(n) \rightarrow \mathbb{CP}^1$ .*

Since  $c_1(E(2)) = 0$ , and  $E(2)$  is simply connected,  $E(2)$  surfaces are exactly K3 surfaces.

*Remark 3.4.* It turns out that a minimal simply connected Elliptic surfaces with sections is always diffeomorphic to an  $E(n)$  surface with  $n > 1$ , see [10].

#### 4. PROOFS OF THEOREMS

*Proof of theorem 1.1.* Let  $X$  be any K3 surface. In the homology group  $H_2(X, \mathbb{Z})$ , take  $s$  and  $f$  to be basis homology classes, generating one of the

3 summands with intersection matrix  $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ . Let  $S$  be a sphere with homology class  $s$  and  $F_1, \dots, F_g$  be  $g$  nonintersecting tori in the homology class  $f$ , all intersecting the sphere  $S$  at exactly one positive transverse intersection. Let  $\Sigma$  be the surface, gotten by resolving all intersections of the union  $S \cup F_1 \cup \dots \cup F_g$ . By a resolution of intersections, we simply mean substituting a local model  $zw = 0$  of an intersection by  $zw = \epsilon$ , where  $\epsilon$  is small. Then  $\Sigma$  is an oriented surface of genus  $g$  in the homology class  $s + gf$ . Applying formulas (2.2), we get  $2I_{\pm}(\Sigma) = \chi(\Sigma) + [\Sigma]^2 = 0$ . So after a small isotopic perturbation, we can make  $\Sigma$  totally real. This way we can construct 3 non-intersecting, non-homologous totally real embeddings of a compact oriented surface of genus  $g$  into  $X$ .

Let us now look at embeddings of nonorientable surfaces. Let  $\Sigma$  be a compact nonorientable surface with Euler characteristics  $\chi(\Sigma) = \chi$ . By a result of Massey [14],  $\Sigma$  can be embedded into  $\mathbb{C}^2$  with  $\chi(N\Sigma) \in \{2\chi - 4, 2\chi, \dots, 4 - 2\chi\}$ . By the same result, this are the only possible normal Euler numbers for such an embedding. Using (2.1), we can achieve  $I(\Sigma)$  to be any one of the numbers in the set

$$N(\chi) := \{3\chi - 4, 3\chi, \dots, 4 - \chi\},$$

by embedding  $\Sigma$  into  $\mathbb{C}^2$ .

**Case 1:**  $\chi \equiv 0 \pmod{4}$ . Since 0 is in the set  $N(\chi)$ , we can embed  $\Sigma$  in a small contractible domain inside any complex surface.

**Case 2:**  $\chi \equiv 2 \pmod{4}$ . Since 2 is in  $N(\chi)$ , we can embed  $\Sigma$  into a small contractible set in  $K3$  with  $I(\Sigma) = 2$ . Let  $S$  be a totally real sphere in  $K3$ , not intersecting  $\Sigma$ , and let  $\Sigma' = \Sigma \# S$ . The connected sum is performed by simply tubing an arc between the surfaces. Then  $\Sigma'$  is again a nonorientable surface with Euler characteristic  $\chi$ . Calculating the algebraic number of complex points using (2.1), we have  $I(\Sigma') = I(\Sigma) + I(S) - 2 = 0$ .

**Case 3:**  $\chi \equiv 3 \pmod{4}$ . Let  $E$  be the exceptional sphere in the blow-up of  $K3$ . Then  $I(E) = \chi(E) + [E]^2 = 1$ . Since  $1 \in N(\chi)$ , let us embed  $\Sigma$  into a small contractible set in  $K3$ , disjoint with  $E$ , so that  $I(\Sigma) = 1$ . Let  $\Sigma' = \Sigma \# E$ . As before,  $\Sigma'$  is nonorientable with Euler characteristic  $\chi$ . We have  $I(\Sigma') = I(\Sigma) + I(E) - 2 = 0$ .

**Case 4:**  $\chi \equiv 1 \pmod{4}$ . Let  $E$  be the exceptional sphere in the blow-up of  $K3$  and  $S$  be the totally real sphere in  $K3$ . We assume they are disjoint. Since  $3 \in N(\chi)$ , let us embed  $\Sigma$  into a small contractible set in  $K3$ , disjoint with  $E$  and  $S$ , so that  $I(\Sigma) = 3$ . Let  $\Sigma' = \Sigma \# E \# S$ .  $\Sigma'$  is nonorientable with Euler characteristic  $\chi$  and  $I(\Sigma') = I(\Sigma) + I(E) + I(S) - 4 = 0$ . This completes the proof.  $\square$

*Proof of theorem 1.2.* Since, as above,  $H_2(E(3), \mathbb{Z})$  group contains a pair of a sphere and a torus, spanning a direct summand in homology with intersection matrix  $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ , we only need to review Cases 3 and 4 in the above proof. So let  $\Sigma$  be a nonorientable surface with  $\chi(\Sigma) = \chi \equiv$

3(mod 4). We can embed  $\Sigma$  into  $E(3)$  having  $I(\Sigma) = 5$ . Let  $S$  be a totally real sphere in  $E(3)$  and let  $S'$  be a disjoint section of  $E(3)$ . We have  $I(S') = \chi(S') + [S']^2 = -1$ . The surface  $\Sigma' = \Sigma \# S \# S'$  is nonorientable with  $\chi(\Sigma') = \chi$  and  $I(\Sigma') = I(\Sigma) + I(S) + I(S') - 4 = 0$ . A small perturbation produces a totally real embedding. If  $\chi(\Sigma) = \chi \equiv 1 \pmod{4}$ , we first embed  $\Sigma$  into  $E(3)$  with  $I(\Sigma) = 3$  and then make  $\Sigma' = \Sigma \# S'$ .  $\square$

*Proof of theorem 1.3.* Let  $S$  be a section of the fibration  $E(m) \rightarrow \mathbb{C}P^1$  and let  $F_1, \dots, F_g$  be  $g$  generic fibers. Let  $\Sigma = S \bigcup_{i=1}^g F_i$  and let  $\tilde{\Sigma}$  be a smooth surface we get from resolving the intersections of  $\Sigma$ . Of course,  $[\Sigma] = [\tilde{\Sigma}]$  and  $\chi(\tilde{\Sigma}) = 2 - 2g$ . By using  $c_1(E(m)) = (2 - m)PD(f)$ , where  $f$  is the homology of a fiber, and applying the formula 2.2, we get  $I(\tilde{\Sigma}) = 0$  and  $I_+(\tilde{\Sigma}) = 2 - m$ . We can thus perturb  $\tilde{\Sigma}$  to a surface, having regular Stein neighborhood basis. Since  $[\tilde{\Sigma}]^2 = ([S] + gf)^2 = -m + 2g$ , we must set  $2g - n = m$ , so that the elements of the Stein neighborhood basis are diffeomorphic to  $D(n, g)$ .  $\square$

*Proof of theorem 1.4.* Let  $\Sigma$  be a nonorientable compact surface with  $\chi(\Sigma) = \chi$ . We use a similar argument as in the proof of Theorem 1.1. As there, we can embed  $\Sigma$  into a small contractible set in  $\mathbb{C}P^2$  so that  $\chi(N\Sigma) \in \{2\chi - 4, 2\chi, \dots, 4 - 2\chi\}$ . Let  $\mathbb{C}P^2 \# m \overline{\mathbb{C}P^2}$  be the blow-up of  $\mathbb{C}P^2$  at  $m$  distinct points, and let  $E_1, \dots, E_m$  be the exceptional spheres. Then  $\Sigma' = \Sigma \# E_1 \cdots \# E_m$  is again a nonorientable surface with Euler characteristic  $\chi$  and  $\chi(N\Sigma') = \chi(N\Sigma) - m$ . For a right choice of  $m$ , we can always achieve that  $n \in \{2\chi - 4, 2\chi, \dots, 4 - 2\chi\} - m$ . We can thus choose an embedding of  $\Sigma$  so that  $\Sigma'$  has  $\chi(N\Sigma') = n$ . Since  $I(\Sigma') = \chi + n \leq 0$ , a perturbation of  $\Sigma'$  has a regular strictly pseudoconvex Stein neighborhood basis of the right topological type.  $\square$

*Remark 4.1.* Instead of a blow-up of  $\mathbb{C}P^2$ , we could also use surfaces  $E(m)$  to put Stein structures on  $\tilde{D}(n, \chi)$ . Instead of taking connected sums with  $m$  exceptional spheres in the blow-up, we take just one connected sum with a section of  $E(m)$ .

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